

## A master equation representation of two-dimensional turbulence

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1993 J. Phys. A: Math. Gen. 26 7563

(<http://iopscience.iop.org/0305-4470/26/24/028>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.68

The article was downloaded on 01/06/2010 at 20:38

Please note that [terms and conditions apply](#).

# A master equation representation of two-dimensional turbulence

Heinz-Peter Breuer and Francesco Petruccione

Albert-Ludwigs-Universität, Fakultät für Physik, Hermann-Herder Strasse 3, W-7800 Freiburg im Breisgau, Federal Republic of Germany

Received 29 April 1993

**Abstract.** A stochastic representation of the turbulent dynamics of the two-dimensional Navier–Stokes equation is developed. This stochastic formulation interprets the fluctuating velocity field as a discrete, multivariate stochastic process which is governed by a master equation. By deriving the Hopf functional equation of statistical fluid mechanics it is shown that this approach yields a complete description of the stochastic properties of the turbulent velocity field. On the basis of the multivariate master equation a stochastic simulation method for the two-dimensional Navier–Stokes equation is constructed. This method is applied to the stochastic simulation of high-Reynolds-number turbulent flows.

## 1. Introduction

One of the challenging problems in computational fluid dynamics is the simulation of fully developed Navier–Stokes turbulence. As far as theories of turbulence are concerned [1–4] numerical investigations may serve as important testing grounds and, furthermore, as a starting point for the development of new ideas. On the other hand, simulations of strongly turbulent fluid flows are interesting on their own since such flows represent excellent examples for systems with many, strongly interacting, degrees of freedom.

Recently a new approach to fluid dynamics has been proposed [5–8]. In this approach the velocity field is regarded as a discrete stochastic process [9, 10] which is governed by a multivariate master equation. The latter leads to a simple stochastic simulation algorithm by which an ensemble of realizations of the multivariate stochastic process is generated. Any physical quantity can then be obtained by evaluating the corresponding ensemble average. This stochastic approach has been illustrated by means of various  $(1+1)$ -dimensional examples from fluid dynamics [6]. In particular, the stability of the stochastic simulation method has been demonstrated by simulating shock-wave and soliton-like solutions of Burgers' equation [5]. Moreover, we have investigated the Burgers' model of turbulence. Employing the Hopf functional equation [11–14] it has been shown [8] that the multivariate master equation leads to a complete description of the stochastic properties of the turbulence. This fact was illustrated by performing some stochastic simulations of the Burgers' turbulence model which yield, for example, the correct behaviour for the energy dissipation rate and the energy spectrum.

It is the purpose of the present paper to generalize these concepts to two space dimensions. That is, we are going to construct a unified stochastic formulation of the dynamics of the two-dimensional incompressible Navier–Stokes equation by means of a multivariate master equation. The latter governs the dynamics of a discrete stochastic

process which corresponds to the fluctuating velocity field of the turbulence. Furthermore, we derive from the master equation the Hopf functional equation for the characteristic functional of statistical hydrodynamics. Thus, our stochastic formulation offers a new approach to the Hopf characteristic functional by stochastic simulation methods.

The master equation formulation developed in this paper should be distinguished from other discrete approaches to computational fluid dynamics as, for example, lattice-gas cellular automata [15] or the lattice Boltzmann equation [16]. Within these approaches the dynamics of the fluid is described on a more microscopic level by means of a simplified version of the microscopic interactions. In contrast, our stochastic representation may be regarded as a mesoscopic approach which describes the fluid on the level of the theory of fluctuating hydrodynamics. This latter point is discussed in [17].

The paper is organized as follows. Having presented in section 2 the basic equations and our treatment of the incompressibility condition, we formulate in section 3 the general theoretical framework of our stochastic interpretation of fluid dynamics. We construct the discrete phase space of the fluid and postulate a discrete multivariate master equation which defines the probabilistic time evolution of the random velocity variables. The master equation is defined within a formalism which draws some analogies to the occupation number formalism of the second quantization in many-body quantum mechanics. This fact is employed in order to derive the general form for the time evolution equation of the expectation value of an arbitrary function of the stochastic variables.

In section 4 the characteristic functional pertaining to our multivariate stochastic process is constructed and shown to obey the Hopf functional equation of statistical hydrodynamics. This means that the master equation defined in section 3 provides a complete description of the whole hierarchy of the  $n$ -point correlation functions of the turbulent field.

Section 5 is devoted to the stochastic simulation method. We derive a stochastic simulation algorithm from our master equation. On the basis of the theoretical investigations of section 4, this algorithm allows us to generate an ensemble of realizations of the turbulent velocity field and to estimate statistical quantities by ensemble averages. The stochastic simulation method is exemplified by performing some simulations of two-dimensional homogeneous turbulence. In particular, the behaviour of the enstrophy dissipation rate and the energy spectrum are discussed.

Finally, in section 6 we summarize the contents of the paper and draw our conclusions.

## 2. Basic equations and treatment of the constraint

Let us first briefly define the basic dynamic equations from which we start our analysis. Throughout this paper we work within the so-called stream function formulation of the two-dimensional Navier–Stokes equation [18]. Introducing the stream function  $\psi(x, t)$  and the vorticity field  $\omega(x, t)$  on the two-dimensional plane with coordinates  $\mathbf{x} \equiv (x, y)$ , the Navier–Stokes equation for an incompressible fluid may be written

$$\frac{\partial \omega}{\partial t} = \nu \Delta \omega + \frac{\partial}{\partial x} \left( \psi \frac{\partial \omega}{\partial y} \right) - \frac{\partial}{\partial y} \left( \psi \frac{\partial \omega}{\partial x} \right). \quad (1)$$

The  $x$  and  $y$  components of the velocity field  $\mathbf{v} \equiv (u, v)$  are given by

$$u = \partial \psi / \partial y \quad v = -\partial \psi / \partial x \quad (2)$$

and the connection between the vorticity field  $\omega = \partial_x v - \partial_y u$  and the stream function  $\psi$  appears as the constraint

$$\Delta\psi + \omega = 0. \quad (3)$$

To be specific, we restrict ourselves in this paper to the most simple boundary conditions. We are seeking for solutions  $(\omega, \psi)$  in the space  $S$  of smooth functions which are periodic on the square  $Q = [0, L] \times [0, L]$  and have zero average

$$\int dx dy \omega = 0 \quad \int dx dy \psi = 0. \quad (4)$$

Note that the first equation in (4) is compatible with the Navier–Stokes equation since it follows from (1) that the total vorticity is conserved and is, therefore, zero for all times if it is zero initially. Furthermore, the second equation in (4) guarantees that the constraint equation (3) can be solved uniquely for  $\psi$ . This follows from the fact that the Laplacian  $\Delta$  is invertible in the space  $S$  of functions defined by the above conditions.

Of course, working in position space the constraint (3) leads to a non-local integral equation for  $\psi$ . Accordingly, the master equation to be constructed below will then contain non-local transitions. However, this difficulty can be circumvented by the following technique. Instead of using the constraint (3),  $\psi$  is regarded as an independent dynamical variable which obeys the following evolution equation

$$\frac{\partial\psi}{\partial t} = \frac{1}{\varepsilon} (\Delta\psi + g\omega) \quad \varepsilon > 0 \quad g = 1 - \varepsilon\nu. \quad (5)$$

Note first, that equation (1) together with equation (5) represent a consistent set of dynamic equations in the space  $S$ . Now, consider  $\varepsilon$  to be a small quantity, i.e. consider the limit  $\varepsilon \rightarrow 0$ . This limit can be studied by means of the general method of elimination of fast variables [19]. This method tells us that for small  $\varepsilon$  the stream function  $\psi$  becomes a fast variable whereas the vorticity  $\omega$  is a slow variable. Furthermore, as is demonstrated by the general method, to leading order the fast variable  $\psi$  becomes a function of the slow variable  $\omega$ . This means that the dynamics is confined to a manifold in the function space  $S$  which is parametrized by the slow variable and which is obtained by setting the expression within the brackets on the right-hand side in (5) equal to zero and by putting  $\varepsilon = 0$  (i.e.  $g = 1$ ). Thus, this constraint manifold is precisely that one given by our original constraint (3).

A necessary condition for the applicability of the method of elimination of fast variables is the following one: for fixed values of the slow variables, each point of the constraint manifold must be an attractor of the dynamic equation for the fast variable [19]. In our case, this condition is obviously fulfilled since the Laplacian  $\Delta$  restricted to the space  $S$  is a negative operator, i.e. has only strictly negative eigenvalues. In an appendix we derive a condition for  $\varepsilon$  which guarantees that  $(\omega, \psi)$  is confined to the neighbourhood of the constraint manifold.

It might be important to note that the three-dimensional Navier–Stokes equation for an incompressible fluid may likewise be obtained from the general equation for a compressible fluid by the method of elimination of fast variables. In this case, the small quantity is the compressibility of the fluid (or the inverse of the velocity of sound) and the constraint manifold is given by  $\text{div } \mathbf{v} = 0$  [19].

### 3. The master equation formulation of statistical Hydrodynamics

We now turn to the construction of a stochastic representation of the two-dimensional Navier–Stokes equation. This will be done within the stream function formalism and starting from the dynamical system defined by equations (1) and (5). The basic idea of our stochastic formulation is to interpret the fields  $\psi$  and  $\omega$  appearing in these partial differential equations as certain multivariate, discrete stochastic processes.

In order to define the latter we first specify the discrete phase space, that is the set of states of the fluid. To this end, we partition the position space, i.e. the square  $Q$ , into small square cells (of area  $\delta l^2$ ) labelled by two integers  $(\lambda, \mu)$ . Thus, we write

$$x_{\lambda\mu} \equiv (x_{\lambda\mu}, y_{\lambda\mu}) = (\lambda \cdot \delta l, \mu \cdot \delta l) \quad (6)$$

for the discrete position vector, where

$$(\lambda, \mu) \in \{0, 1, \dots, M\} \times \{0, 1, \dots, M\} \quad (7)$$

and  $\delta l = L/(M + 1)$  denotes a mesoscopic length scale. Moreover, the values of the stream function and the vorticity are discretized by defining mesoscopic scales  $\delta\psi$  and  $\delta\omega$ . This means that  $\psi$  and  $\omega$  are measured in integer multiples of  $\delta\psi$  and  $\delta\omega$  respectively. In other words, the fields  $\psi$  and  $\omega$  are represented, on a mesoscopic level, by two  $(M + 1)^2$ -dimensional arrays of integers:

$$N_\omega := \{N_\omega^{\lambda\mu}\} \quad N_\psi := \{N_\psi^{\lambda\mu}\}. \quad (8)$$

For convenience we choose

$$\delta\psi = \delta l^2 \delta\omega. \quad (9)$$

This choice is motivated by the constraint (3) which provides the connection between the stream function and the vorticity field. Fourier-transforming the constraint (3) we obtain in an obvious notation

$$k^2 \psi_k = \omega_k.$$

Due to our discretization of position space into cells of size  $\delta l^2$ , the maximal wavenumber  $|k|$  is of order  $1/\delta l^2$  and, therefore, the corresponding Fourier modes of  $\psi$  are by a factor of  $\delta l^2$  smaller than the Fourier modes of  $\omega$ . Since according to (9)  $\delta\psi$  is smaller than  $\delta\omega$  by the same factor, it is guaranteed that these high wavenumber components are resolved within the discretization of the values of the stream function.

The discrete phase space  $\Gamma$  of the fluid which replaces the function space  $S$  introduced in section 2 may now be written as

$$\Gamma = \{(N_\omega, N_\psi) \mid N_\omega^{\lambda\mu} \in \mathbb{Z}, N_\psi^{\lambda\mu} \in \mathbb{Z}\} \quad (10)$$

where  $\mathbb{Z}$  denotes the set of integers.

Now, the stochastic dynamics comes in by regarding  $N_\omega^{\lambda\mu}$  and  $N_\psi^{\lambda\mu}$  as time-dependent random numbers, i.e. we consider  $(N_\omega, N_\psi)$  as a  $2(M + 1)^2$ -variate stochastic process. Correspondingly, we introduce the joint probability distribution

$$P = P(N_\omega, N_\psi; t) \quad (11)$$

giving the probability of finding at time  $t$  the two sets of numbers  $N_\omega$  and  $N_\psi$ . Of course,  $P$  is assumed to be normalized.

$$\sum_{N_\omega N_\psi} P(N_\omega, N_\psi; t) = 1 \tag{12}$$

where  $\sum_{N_\omega N_\psi}$  denotes a  $2(M+1)^2$ -fold sum over all integers  $N_\omega^{\lambda\mu}, N_\psi^{\lambda\mu}$ . With the help of the joint probability distribution  $P$  expectation values of arbitrary functions  $\mathcal{F} = \mathcal{F}(N_\omega, N_\psi)$  of the stochastic variables can be calculated according to

$$\langle \mathcal{F} \rangle := \sum_{N_\omega N_\psi} \mathcal{F}(N_\omega, N_\psi) P(N_\omega, N_\psi; t). \tag{13}$$

We now postulate that the stochastic process  $(N_\omega, N_\psi)$  is Markovian. This postulate is motivated by the fact that the stochastic process defined by the statistical formulation of hydrodynamics is also Markovian [8]. Under these conditions the stochastic process is completely defined by a master equation for the joint probability distribution  $P$ , once an initial distribution has been given. This master equation may be written in the compact form

$$\partial P / \partial t = \mathcal{A}P. \tag{14}$$

Here, the time evolution operator  $\mathcal{A}$  represents a linear operator which acts (to the right) on functions of the stochastic variables. Adopting this general form for the master equation we now define an appropriate operator  $\mathcal{A}$ .

As has been mentioned at the beginning of this section the basic idea on which our stochastic interpretation is based is the following one: the fields  $\omega$  and  $\psi$  appearing in the Navier–Stokes and the constraint equation are replaced by the multivariate stochastic process  $(N_\omega, N_\psi)$ . To make this idea more precise, recall that within the usual statistical description of the turbulence problem the vorticity field and the stream function are considered as fluctuating fields. Following the Reynolds averaging procedure one obtains an infinite set of coupled equations for the set of  $n$ th-order moments of these fields. For example, the equations for the first moments of the stream function and the vorticity field take the form

$$\begin{aligned} \frac{\partial}{\partial t} \langle \omega \rangle &= \nu \Delta \langle \omega \rangle + \left\langle \frac{\partial}{\partial x} \left( \psi \frac{\partial \omega}{\partial y} \right) - \frac{\partial}{\partial y} \left( \psi \frac{\partial \omega}{\partial x} \right) \right\rangle \\ \frac{\partial}{\partial t} \langle \psi \rangle &= \frac{1}{\varepsilon} (\Delta \langle \psi \rangle + g \langle \omega \rangle). \end{aligned} \tag{15}$$

Within our discrete formulation the random fields  $\omega$  and  $\psi$  are replaced by the random quantities defined by

$$\omega_{\lambda\mu} := \delta \omega N_\omega^{\lambda\mu} \quad \psi_{\lambda\mu} := \delta \psi N_\psi^{\lambda\mu}. \tag{16}$$

Now, our aim is to define a time evolution operator  $\mathcal{A}$  and, thereby, a master equation in such a way that the time evolution of the moments of the stochastic processes (16) is governed by a discretized form of the coupled system of dynamic moment equations which completely characterizes the statistical properties of the turbulence. In particular, we demand that the first moments obey the following discretized version of the equations (15):

$$\begin{aligned} \frac{\partial}{\partial t} \langle \omega_{\lambda\mu} \rangle &= \nu D \langle \omega_{\lambda\mu} \rangle + \langle d_1(\psi_{\lambda\mu} d_2 \omega_{\lambda\mu}) - d_2(\psi_{\lambda\mu} d_1 \omega_{\lambda\mu}) \rangle \\ \frac{\partial}{\partial t} \langle \psi_{\lambda\mu} \rangle &= \frac{1}{\varepsilon} (D \langle \psi_{\lambda\mu} \rangle + g \langle \omega_{\lambda\mu} \rangle). \end{aligned} \tag{17}$$

For the sake of a compact notation we have introduced here the discrete operators  $d_1$ ,  $d_2$  and  $D$  which replace the partial differential operators  $\partial/\partial x$ ,  $\partial/\partial y$  and the Laplacian  $\Delta$  and which are defined by

$$\begin{aligned} d_1 f_{\lambda\mu} &:= \frac{f_{\lambda+1,\mu} - f_{\lambda-1,\mu}}{2\delta l} \\ d_2 f_{\lambda\mu} &:= \frac{f_{\lambda,\mu+1} - f_{\lambda,\mu-1}}{2\delta l} \\ D f_{\lambda\mu} &:= \frac{f_{\lambda+1,\mu} + f_{\lambda-1,\mu} + f_{\lambda,\mu+1} + f_{\lambda,\mu-1} - 4f_{\lambda\mu}}{\delta l^2} \end{aligned} \quad (18)$$

where  $f_{\lambda\mu}$  denotes an arbitrary function on the discrete grid introduced above.

An appropriate time evolution operator  $\mathcal{A}$  which fulfils the requirement just explained can be decomposed into three parts,

$$\mathcal{A} = \mathcal{A}_d + \mathcal{A}_c + \mathcal{A}_p. \quad (19)$$

Each of these three operators corresponds to a certain part of the right-hand sides of differential equations (1) and (5). The structure of these operators is conveniently described with the help of the following shift operators which act on functions  $\mathcal{F}(N_\omega, N_\psi)$  of the stochastic variables:

$$\begin{aligned} H_{\lambda\mu}^{\pm 1} \mathcal{F}(\dots, N_\omega^{\lambda\mu}, \dots) &:= \mathcal{F}(\dots, N_\omega^{\lambda\mu} \pm 1, \dots) \\ F_{\lambda\mu}^{\pm 1} \mathcal{F}(\dots, N_\psi^{\lambda\mu}, \dots) &:= \mathcal{F}(\dots, N_\psi^{\lambda\mu} \pm 1, \dots). \end{aligned} \quad (20)$$

Thus, these operators change the random integers by  $\pm 1$  and therefore induce the most simple transitions of the state of the fluid. Employing these shift operators the operator  $\mathcal{A}_d$  which describes the viscous diffusion term of the vorticity equation (1) may be written in the form

$$\begin{aligned} \mathcal{A}_d &= \frac{\nu}{\delta l^2} \sum_{\lambda\mu} (H_{\lambda-1,\mu}^{-1} H_{\lambda\mu} - \mathbf{1})({}^+ N_\omega^{\lambda\mu} - {}^- N_\omega^{\lambda-1,\mu}) + (H_{\lambda+1,\mu}^{-1} H_{\lambda\mu} - \mathbf{1})({}^+ N_\omega^{\lambda\mu} - {}^- N_\omega^{\lambda+1,\mu}) \\ &\quad + (H_{\lambda,\mu-1}^{-1} H_{\lambda\mu} - \mathbf{1})({}^+ N_\omega^{\lambda\mu} - {}^- N_\omega^{\lambda,\mu-1}) \\ &\quad + (H_{\lambda,\mu+1}^{-1} H_{\lambda\mu} - \mathbf{1})({}^+ N_\omega^{\lambda\mu} - {}^- N_\omega^{\lambda,\mu+1}) \end{aligned} \quad (21)$$

where  $\mathbf{1}$  denotes the identity operator. The nonlinear convection term of the vorticity equation is represented by the operator

$$\begin{aligned} \mathcal{A}_c &= \frac{\delta\psi}{4\delta l^2} \sum_{\lambda\mu} (H_{\lambda+1,\mu}^{-1} H_{\lambda\mu} - \mathbf{1})({}^+ \mathcal{N}^{\lambda\mu} + {}^+ \mathcal{N}^{\lambda+1,\mu}) - (H_{\lambda+1,\mu} H_{\lambda\mu}^{-1} - \mathbf{1})({}^- \mathcal{N}^{\lambda\mu} + {}^- \mathcal{N}^{\lambda+1,\mu}) \\ &\quad + (H_{\lambda,\mu+1}^{-1} H_{\lambda\mu} - \mathbf{1})({}^+ \mathcal{M}^{\lambda\mu} + {}^+ \mathcal{M}^{\lambda,\mu+1}) \\ &\quad - (H_{\lambda,\mu+1} H_{\lambda\mu}^{-1} - \mathbf{1})({}^- \mathcal{M}^{\lambda\mu} + {}^- \mathcal{M}^{\lambda,\mu+1}) \end{aligned} \quad (22)$$

whereas the constraint equation is modelled by the operator

$$\mathcal{A}_p = \frac{1}{\varepsilon} \sum_{\lambda\mu} (F_{\lambda\mu}^{-1} - \mathbf{1}) {}^+ W^{\lambda\mu} - (F_{\lambda\mu}^{+1} - \mathbf{1}) {}^- W^{\lambda\mu}. \quad (23)$$

Writing these expressions we have furthermore introduced the quantities

$$\begin{aligned}
 \mathcal{N}^{\lambda\mu} &:= -N_{\psi}^{\lambda\mu} (N_{\omega}^{\lambda,\mu+1} - N_{\omega}^{\lambda,\mu-1}) \\
 \mathcal{M}^{\lambda\mu} &:= +N_{\psi}^{\lambda\mu} (N_{\omega}^{\lambda+1,\mu} - N_{\omega}^{\lambda-1,\mu}) \\
 \mathcal{W}^{\lambda\mu} &:= DN_{\psi}^{\lambda\mu} + g \frac{\delta\omega}{\delta\psi} N_{\omega}^{\lambda\mu}.
 \end{aligned}
 \tag{24}$$

Moreover, we define the positive part  ${}^+\mathcal{F}$  and negative part  ${}^-\mathcal{F}$  of a function  $\mathcal{F}$  of the stochastic variables by the relations:

$$\mathcal{F} = {}^+\mathcal{F} + {}^-\mathcal{F} \quad |\mathcal{F}| = {}^+\mathcal{F} - {}^-\mathcal{F}.
 \tag{25}$$

The multivariate master equation (14) together with (19) and these definitions completely defines our stochastic process  $(N_{\omega}, N_{\psi})$  and constitutes the starting point of the theoretical and numerical investigations in this paper. Note that the master equation specifies all possible transitions among the states of the phase space  $\Gamma$  and gives the corresponding transition rates. These transitions are described in terms of the shift operators. For example,  $H_{\lambda\mu}^{+1}$  raises the stochastic variable  $N_{\omega}^{\lambda\mu}$  by 1 and  $H_{\lambda\mu}^{-1}$  lowers this variable by 1. Correspondingly, the term  $H_{\lambda+1,\mu}^{-1} H_{\lambda\mu}$ , for example, describes the jump of a ‘quantum’  $\delta\omega$  of vorticity from cell  $(\lambda, \mu)$  to cell  $(\lambda+1, \mu)$ . All transitions which appear in our master equation are written in terms of such one-particle jumps.

The structure of the time evolution operators given above might look rather complicated. The reader who is unfamiliar with the operator notation used here, may consult [6] which contains a more detailed exposition of the above notation. Moreover, we describe in section 5 how a typical realization of the multivariate stochastic process can be obtained. This might be helpful as well in order to understand the meaning of the master equation.

Rather than giving a heuristic construction of our master equation (this has been done for the one-dimensional Burgers’ equation in previous work [6]) we will demonstrate in the next section that this master equation fulfils our requirement stated above, i.e. that it correctly describes the whole hierarchy of the moment equations which are characteristic for the turbulence problem. To this end, we derive in the following some general properties of our master equation which serve as a starting point of the considerations in the next section.

It should be clear that our formulation of the master equation resembles the occupation number formalism of many-body quantum mechanics. This analogy is also reflected in the general structure of the time evolution equation of an arbitrary function  $\mathcal{F}(N_{\omega}, N_{\psi})$  of the stochastic variables. That is, regarding  $\mathcal{F}$  as a multiplicative operator we now show that the time derivative of the expectation value of  $\mathcal{F}$  is given by the expectation value of the commutator,

$$\frac{\partial}{\partial t} \langle \mathcal{F} \rangle = \langle [\mathcal{F}, \mathcal{A}] \rangle.
 \tag{26}$$

In order to prove this equation we first observe that, for example,

$$\langle (H_{\lambda\pm 1, \mu\pm 1}^{-1} H_{\lambda\mu} - 1) \mathcal{F} \rangle = 0
 \tag{27}$$

as is easily seen by taking into account that the expectation value involves a multiple sum over all stochastic variables and by shifting the summation indices appropriately. Analogous



equations are valid for any other product of shift operators appearing in our master equation. Thus, we immediately conclude from the structure of our master equation that the following equation holds

$$\langle \mathcal{A}\mathcal{F} \rangle = 0. \tag{28}$$

This property of the time evolution operator guarantees that (14) is, in fact, a master equation. Employing the definition of the expectation value (13) and the master equation (14) and invoking (28) we find

$$\frac{\partial}{\partial t} \langle \mathcal{F} \rangle = \sum_{N_\omega N_\psi} \mathcal{F} \frac{\partial}{\partial t} P = \sum_{N_\omega N_\psi} \mathcal{F} \mathcal{A}P = \sum_{N_\omega N_\psi} [\mathcal{F}, \mathcal{A}]P = \langle [\mathcal{F}, \mathcal{A}] \rangle. \tag{29}$$

Equation (26) offers a convenient way of determining the time evolution equations of arbitrary moments of the stochastic variables. This is the subject of the next section.

#### 4. Derivation of the Hopf functional equation

It should be clear that the multivariate master equation (14) is nonlinear. This is due to the fact that the transition rates corresponding to the convection operator  $\mathcal{A}_c$  depend nonlinearly upon the stochastic variables (see equations (22) and (24)). Thus, the moment equations which follow from our master equation are not closed and form an infinite system of coupled equations. We will demonstrate in this section that this system of moment equations is precisely that one which is known from the general theory of turbulence.

In order to derive the dynamic moment equations corresponding to our master equation one may, of course, start from equation (26) using the various moments for the function  $\mathcal{F}$ . However, it is much more convenient to work with the multivariate characteristic function

$$M(\alpha, \beta, t) := \langle \mathcal{G}(\alpha, \beta; N_\omega, N_\psi) \rangle \tag{30}$$

where the stochastic function  $\mathcal{G}$  is defined by

$$\begin{aligned} \mathcal{G}(\alpha, \beta; N_\omega, N_\psi) &:= \exp \left( i\delta l^2 \delta\omega \sum_{\lambda\mu} \alpha_{\lambda\mu} N_\omega^{\lambda\mu} + i\delta l^2 \delta\psi \sum_{\lambda\mu} \beta_{\lambda\mu} N_\psi^{\lambda\mu} \right) \\ &= \exp \left( i\delta l^2 \sum_{\lambda\mu} \alpha_{\lambda\mu} \omega_{\lambda\mu} + i\delta l^2 \sum_{\lambda\mu} \beta_{\lambda\mu} \psi_{\lambda\mu} \right). \end{aligned} \tag{31}$$

In the following we will derive the equation of motion for this characteristic function  $M(\alpha, \beta, t)$ . From this time evolution equation the moment equations can then be obtained simply by differentiating with respect to  $\alpha_{\lambda\mu}$  and  $\beta_{\lambda\mu}$ . Thus, the characteristic function contains all dynamic properties of the  $n$ -point (equal-time) correlation functions.

To start with, we invoke equation (26) to obtain

$$\frac{\partial M}{\partial t} = \langle [\mathcal{G}, \mathcal{A}] \rangle = \langle [\mathcal{G}, \mathcal{A}_d] \rangle + \langle [\mathcal{G}, \mathcal{A}_c] \rangle + \langle [\mathcal{G}, \mathcal{A}_p] \rangle. \tag{32}$$

The commutators appearing in the above equation can be evaluated by means of the following commutation relations which are easily verified by applying the operators on both sides to an arbitrary function of the stochastic variables:

$$\begin{aligned}
 [G, H_{\lambda\pm 1, \mu}^{-1} H_{\lambda\mu}] &= H_{\lambda\pm 1, \mu}^{-1} H_{\lambda\mu} (\exp(i\delta l^2 \delta\omega\{\alpha_{\lambda\pm 1, \mu} - \alpha_{\lambda\mu}\}) - 1)G \\
 [G, H_{\lambda, \mu\pm 1}^{-1} H_{\lambda\mu}] &= H_{\lambda, \mu\pm 1}^{-1} H_{\lambda\mu} (\exp(i\delta l^2 \delta\omega\{\alpha_{\lambda, \mu\pm 1} - \alpha_{\lambda\mu}\}) - 1)G \\
 [G, H_{\lambda+1, \mu} H_{\lambda\mu}^{-1}] &= H_{\lambda+1, \mu} H_{\lambda\mu}^{-1} (\exp(i\delta l^2 \delta\omega\{\alpha_{\lambda, \mu} - \alpha_{\lambda+1, \mu}\}) - 1)G \\
 [G, H_{\lambda, \mu+1} H_{\lambda\mu}^{-1}] &= H_{\lambda, \mu+1} H_{\lambda\mu}^{-1} (\exp(i\delta l^2 \delta\omega\{\alpha_{\lambda, \mu} - \alpha_{\lambda, \mu+1}\}) - 1)G \\
 [G, F_{\lambda\mu}^{\pm 1}] &= F_{\lambda\mu}^{\pm 1} (\exp(\mp i\delta l^2 \delta\psi\beta_{\lambda, \mu}) - 1)G.
 \end{aligned}
 \tag{33}$$

Using these relations we find

$$\begin{aligned}
 \langle [G, \mathcal{A}_d] \rangle &= \frac{\nu}{\delta l^2} \sum_{\lambda\mu} \langle (\exp(i\delta l^2 \delta\omega[\alpha_{\lambda-1, \mu} - \alpha_{\lambda\mu}]) - 1) (+N_{\omega}^{\lambda\mu} - -N_{\omega}^{\lambda-1, \mu})G \rangle \\
 &\quad + \langle (\exp(i\delta l^2 \delta\omega[\alpha_{\lambda+1, \mu} - \alpha_{\lambda\mu}]) - 1) (+N_{\omega}^{\lambda\mu} - -N_{\omega}^{\lambda+1, \mu})G \rangle \\
 &\quad + \langle (\exp(i\delta l^2 \delta\omega[\alpha_{\lambda, \mu-1} - \alpha_{\lambda\mu}]) - 1) (+N_{\omega}^{\lambda\mu} - -N_{\omega}^{\lambda, \mu-1})G \rangle \\
 &\quad + \langle (\exp(i\delta l^2 \delta\omega[\alpha_{\lambda, \mu+1} - \alpha_{\lambda\mu}]) - 1) (+N_{\omega}^{\lambda\mu} - -N_{\omega}^{\lambda, \mu+1})G \rangle \\
 \langle [G, \mathcal{A}_c] \rangle &= \frac{\delta\psi}{4\delta l^2} \sum_{\lambda\mu} \langle (\exp(i\delta l^2 \delta\omega[\alpha_{\lambda+1, \mu} - \alpha_{\lambda\mu}]) - 1) (+\mathcal{N}^{\lambda\mu} + +\mathcal{N}^{\lambda+1, \mu})G \rangle \\
 &\quad - \langle (\exp(i\delta l^2 \delta\omega[\alpha_{\lambda, \mu} - \alpha_{\lambda+1, \mu}]) - 1) (-\mathcal{N}^{\lambda\mu} + -\mathcal{N}^{\lambda+1, \mu})G \rangle \\
 &\quad + \langle (\exp(i\delta l^2 \delta\omega[\alpha_{\lambda, \mu+1} - \alpha_{\lambda\mu}]) - 1) (+\mathcal{M}^{\lambda\mu} + +\mathcal{M}^{\lambda, \mu+1})G \rangle \\
 &\quad - \langle (\exp(i\delta l^2 \delta\omega[\alpha_{\lambda, \mu} - \alpha_{\lambda, \mu+1}]) - 1) (-\mathcal{M}^{\lambda\mu} + -\mathcal{M}^{\lambda, \mu+1})G \rangle \\
 \langle [G, \mathcal{A}_p] \rangle &= \frac{1}{\varepsilon} \sum_{\lambda\mu} \langle (\exp(+i\delta l^2 \delta\psi\beta_{\lambda\mu}) - 1) \\
 &\quad - \langle (\exp(-i\delta l^2 \delta\psi\beta_{\lambda\mu}) - 1).
 \end{aligned}
 \tag{34}$$

Employing these equations together with the time evolution equation for the multivariate characteristic function  $M$ , equation (32), one obtains the time evolution equations for arbitrary moments of the stochastic variables  $(N_{\omega}, N_{\psi})$ . For example, differentiating both sides of (32) with respect to  $\alpha_{\lambda\mu}$  and using the identity

$$\langle N_{\omega}^{\lambda\mu} N_{\omega}^{\lambda'\mu'} \dots \rangle = \left( \frac{1}{i\delta l^2 \delta\omega} \frac{\partial}{\partial \alpha_{\lambda\mu}} \right) \left( \frac{1}{i\delta l^2 \delta\omega} \frac{\partial}{\partial \alpha_{\lambda'\mu'}} \right) \dots M(\alpha, \beta, t) \Big|_{\alpha=\beta=0}
 \tag{35}$$

yields the dynamic equations for the moments of the random vorticity. In particular, for

$$\langle \omega_{\lambda\mu} \rangle = \frac{1}{i\delta l^2} \frac{\partial M}{\partial \alpha_{\lambda\mu}} \Big|_{\alpha=\beta=0} \quad \langle \psi_{\lambda\mu} \rangle = \frac{1}{i\delta l^2} \frac{\partial M}{\partial \beta_{\lambda\mu}} \Big|_{\alpha=\beta=0}
 \tag{36}$$

one finds equations (17).

Of course, one could go on deriving the corresponding equations for the higher moments. However, we are interested in a comparison of these moment equations with the moment hierarchy which is known from the general statistical theory of turbulence and which involves partial differential equations for the moments of continuous fields. It is therefore more convenient to adopt the following strategy.

First, note that the time evolution equation (32) for the characteristic function  $M$  depends upon the mesoscopic scales  $\delta\omega$  and  $\delta l$  (note that  $\delta\psi$  is fixed by (9)) which have been used in order to discretize the phase space. In order to see the meaning of the vorticity scale  $\delta\omega$  note that in accordance with equation (16) the random integers  $N_\omega^{\lambda\mu}$  scale with  $\delta\omega^{-1}$ , that is for a fixed value of  $\omega_{\lambda\mu}$  these integers become arbitrary large if the limit of continuous vorticity is taken, i.e. if we let  $\delta\omega \rightarrow 0$ . Thus, we expect that the random fields  $\omega_{\lambda\mu}$  and  $\psi_{\lambda\mu}$  are independent of the vorticity scale  $\delta\omega$  to leading order in the continuum limit. Therefore, we first perform an expansion with respect to  $\delta\omega$ . Then, in a second step, the limit of continuous space is investigated, that is, we let  $\delta l \rightarrow 0$ . It will turn out that the resulting continuous form of the time evolution equation for  $M$  is identical to the Hopf functional equation which serves as a complete description of the statistical properties of the turbulence.

It is important to note that this procedure amounts to an asymptotic expansion of the equation of motion for  $M$  which is justified by the fact that the dominant contribution of this expansion, i.e. the Hopf functional equation, does not depend upon  $\delta\omega$  and  $\delta l$  (see below). This fact is achieved by an appropriate scaling of the stochastic variables in the stochastic function  $\mathcal{G}$  (see equation (31)). The same technique has been used in the context of asymptotic expansions of reaction-diffusion systems [20].

To begin with, we now perform two steps. First, the exponentials occurring in (34) are expanded in powers of  $\delta l^2 \delta\omega$  and  $\delta l^2 \delta\psi$  respectively, including terms of second order. Second, we invoke the fact that the stochastic variables within the angular brackets can be expressed by partial derivatives which act on the characteristic function  $M = \langle \mathcal{G} \rangle$ , i.e. we replace

$$N_\omega^{\lambda\mu} \rightarrow \frac{1}{i\delta l^2 \delta\omega} \frac{\partial}{\partial \alpha_{\lambda\mu}} \quad N_\psi^{\lambda\mu} \rightarrow \frac{1}{i\delta l^2 \delta\psi} \frac{\partial}{\partial \beta_{\lambda\mu}}. \tag{37}$$

We then obtain, including terms of order  $\delta l^2 \delta\omega$  and neglecting terms of order  $\delta\omega \delta l^3$ ,

$$\begin{aligned} \langle [\mathcal{G}, \mathcal{A}_d] \rangle &= \nu \sum_{\lambda\mu} \delta l^2 (D\alpha_{\lambda\mu}) \frac{1}{\delta l^2} \frac{\partial}{\partial \alpha_{\lambda\mu}} M \\ &\quad - \nu \delta\omega \delta l^2 \sum_{\lambda\mu} \delta l^2 \{ (d_1 \alpha_{\lambda,\mu})^2 + (d_2 \alpha_{\lambda,\mu})^2 \} \left| \frac{1}{\delta l^2} \frac{\partial}{\partial \alpha_{\lambda\mu}} \right| M \\ \langle [\mathcal{G}, \mathcal{A}_c] \rangle &= i \sum_{\lambda\mu} \delta l^2 \left\{ (d_1 \alpha_{\lambda\mu}) \left( \frac{1}{\delta l^2} \frac{\partial}{\partial \beta_{\lambda\mu}} \right) d_2 \left( \frac{1}{\delta l^2} \frac{\partial}{\partial \alpha_{\lambda\mu}} \right) \right. \\ &\quad \left. - (d_2 \alpha_{\lambda\mu}) \left( \frac{1}{\delta l^2} \frac{\partial}{\partial \beta_{\lambda\mu}} \right) d_1 \left( \frac{1}{\delta l^2} \frac{\partial}{\partial \alpha_{\lambda\mu}} \right) \right\} M \\ \langle [\mathcal{G}, \mathcal{A}_p] \rangle &= \frac{1}{\varepsilon} \sum_{\lambda\mu} \delta l^2 \left\{ (D\beta_{\lambda\mu}) \frac{1}{\delta l^2} \frac{\partial}{\partial \beta_{\lambda\mu}} + g\beta_{\lambda\mu} \frac{1}{\delta l^2} \frac{\partial}{\partial \alpha_{\lambda\mu}} \right\} M. \end{aligned} \tag{38}$$

Now, in the continuum limit  $\delta l \rightarrow 0$  of the time evolution equation (32) the multivariate characteristic function  $M(\alpha, \beta, t)$  becomes a (time-dependent) functional

$M[\alpha, \beta, t]$  in the space of functions  $\alpha(x)$  and  $\beta(x)$  whereas partial derivatives turn into functional derivatives,

$$\frac{1}{\delta l^2} \frac{\partial}{\partial \alpha_{\lambda\mu}} \rightarrow \frac{\delta}{\delta \alpha(x)} \quad \frac{1}{\delta l^2} \frac{\partial}{\partial \beta_{\lambda\mu}} \rightarrow \frac{\delta}{\delta \beta(x)} \tag{39}$$

and sums over  $(\lambda, \mu)$  have to be replaced by integrals,

$$\sum_{\lambda\mu} \delta l^2 \rightarrow \int dx dy. \tag{40}$$

It is then easy to infer from (38) the continuous version of equation (32). By an additional integration by parts we finally arrive at

$$\begin{aligned} \frac{\partial}{\partial t} M[\alpha(x), \beta(x), t] = & i \int dx dy \alpha \left( v \Delta \frac{\delta}{i \delta \alpha} + \left[ \frac{\partial}{\partial x} \frac{\delta}{i \delta \beta} \frac{\partial}{\partial y} \frac{\delta}{i \delta \alpha} - \frac{\partial}{\partial y} \frac{\delta}{i \delta \beta} \frac{\partial}{\partial x} \frac{\delta}{i \delta \alpha} \right] \right) M \\ & + i \int dx dy \beta \frac{1}{\varepsilon} \left( \Delta \frac{\delta}{i \delta \beta} + g \frac{\delta}{i \delta \alpha} \right) M - v \delta \omega \delta l^2 \int dx dy |\nabla \alpha|^2 \left| \frac{\delta}{i \delta \alpha} \right| M. \end{aligned} \tag{41}$$

This is the equation of motion for the characteristic functional  $M[\alpha, \beta, t]$  which has been derived from our master equation for the multivariate stochastic process  $(N_\omega, N_\psi)$  by an asymptotic expansion around the continuum limit of the latter. As is easily seen, the first two lines of equation (41) are identical to the well-known Hopf functional equation [11–14] corresponding to a statistical ensemble which evolves according to the equations of motion (1) and (5).

Recall that the Hopf functional equation yields a complete formulation of the statistical problem of the turbulence. Once an appropriate solution is known, arbitrary correlation functions of the vorticity and stream function can be obtained from functional derivatives of the characteristic functional  $M$ . For example, we have for the two-point vorticity correlation

$$\langle \omega(x_1, t) \omega(x_2, t) \rangle = - \frac{\delta^2}{\delta \alpha(x_2) \delta \alpha(x_1)} M[\alpha(x), \beta(x), t] \Big|_{\alpha=\beta=0}. \tag{42}$$

The third line in equation (41) represents a functional

$$C[\alpha, \beta] := -v \delta \omega \delta l^2 \int dx dy |\nabla \alpha|^2 \left| \frac{\delta}{i \delta \alpha} \right| M \tag{43}$$

which vanishes in the continuum limit  $\delta \omega \rightarrow 0$ . Thus we conclude that within this continuum limit the stochastic process defined by the multivariate master equation yields a complete representation of the system of equations of motion for the  $n$ -point correlation functions of the turbulence problem.

In order to see the meaning of the additional term (43) in our Hopf equation (41) we now derive the equation for the two-point vorticity correlation (42). The latter is obtained by functionally differentiating the Hopf equation (41) twice and by using equation (42). On Fourier transforming the resulting equation and assuming spatial homogeneity we then find

$$\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) \langle \omega_k^* \omega_k \rangle = W_k + 2\nu k^2 \sigma^2 \tag{44}$$

where the Fourier transform of a function  $f(x)$  is defined by

$$f_k := \int dx dy f(x) \exp(-ikx) \quad (45)$$

and  $W_k$  denotes the vorticity transfer function defined by

$$W_k := \frac{1}{L^2} \sum_q (k_y q_x - k_x q_y) \langle \omega_k^* \psi_q \omega_{k-q} \rangle + \text{CC} \quad (46)$$

(CC means that the complex conjugated term is to be added) and, finally,

$$\sigma^2 := \tilde{\omega} \delta \omega \delta l^2 L^2 \quad \tilde{\omega} := \langle |\omega| \rangle. \quad (47)$$

As is to be expected from our general discussion, equation (44) represents the well-known equation of motion for the vorticity correlation function if the continuum limit is taken and, thus, the last term vanishes. This last term in equation (44) is the Fourier transform of

$$-\frac{\delta^2}{\delta \alpha(x_2) \delta \alpha(x_1)} C[\alpha, \beta] \Big|_{\alpha=\beta=0} = -2\nu L^{-2} \nabla_1 \sigma^2 \nabla_1 \delta(x_2 - x_1). \quad (48)$$

This term represents the correlation function of a random vorticity stress and its presence in equation (44) is to be traced to the discrete nature of the stochastic variables ( $N_\omega, N_\psi$ ). In order to see the effect of this random vorticity stress we consider the stationary solution of equation (44) which is given by

$$\langle \omega_k^* \omega_k \rangle_{\text{stat}} = \sigma^2. \quad (49)$$

Thus we see that in the stationary state the random vorticity stress (48) gives rise to an equipartition of vorticity among the Fourier modes. It might be interesting to note that the stationary state given by equation (49) may be identified with the equilibrium state of a canonical ensemble which is based on the enstrophy as a constant of motion. Introducing the corresponding enstrophy temperature  $T$  [3, 4] we find

$$\frac{1}{2} \tilde{\omega} \delta \omega = f k_B T \quad (50)$$

where  $f = (L/\delta l)^2$  denotes the number of degrees of freedom and  $k_B$  the Boltzmann constant. Thus, the quantity  $\delta \omega$  which has been introduced in order to discretize the vorticity is proportional to the enstrophy temperature  $T$  and the number of degrees of freedom. Therefore, the continuum limit  $\delta \omega \rightarrow 0$  may be interpreted as the limit of zero enstrophy temperature.

It is important to note the expansion leading to the Hopf functional equation provides an explicit expression for the random vorticity stress which appears as the last term in equation (44) for the two-point correlation function. We now demonstrate that the vorticity field  $\omega$  may be decomposed into two parts,

$$\omega_k = \hat{\omega}_k + \eta_k \quad (51)$$

in such a way that the first part which is denoted by  $\hat{\omega}$  obeys the equation

$$\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) \langle \hat{\omega}_k^* \hat{\omega}_k \rangle = \hat{W}_k \quad (52)$$

where  $\hat{W}_k$  is defined as in equation (46) with  $\omega_k$  replaced by  $\hat{\omega}_k$ . Thus, in accordance with our above discussion  $\hat{\omega}$  represents the vorticity field at zero enstrophy temperature. The second part  $\eta_k$  in equation (51) denotes a random field which is statistically independent from the vorticity and which obeys

$$\langle \eta_k \rangle = 0 \quad \langle \eta_k^* \eta_k \rangle = g_k. \quad (53)$$

Consequently, we have

$$\langle \omega_k^* \omega_k \rangle = \langle \hat{\omega}_k^* \hat{\omega}_k \rangle + g_k. \quad (54)$$

Inserting equation (54) into equation (44) and choosing the correlation function of the random field  $\eta_k$  to be

$$g_k(t) = 2\nu k^2 \int_0^t d\tau \sigma^2(\tau) \exp\{-2\nu k^2(t - \tau)\} \quad (55)$$

now yields equation (52). Thus we conclude that the noise part in equation (44) induced by the random vorticity stress is removed by the simple transformation (54) which may therefore be used in order to separate uniquely the zero temperature field from the random vorticity governed by our master equation.

Once again we emphasize that the random vorticity stress is known explicitly from the expansion leading to the Hopf equation. Therefore, having performed a stochastic simulation of our master equation (see the next section) this random stress can be determined from the simulation data by means of equation (48). Employing this information it is possible to obtain the zero temperature field  $\hat{\omega}$  from the simulation data by means of the above transformation; in particular, the zero temperature two-point correlation function may be obtained from equation (54). This fact will be used in the next section when performing some stochastic simulations of the two-dimensional turbulence.

## 5. Stochastic simulations of two-dimensional turbulence

Having presented the theoretical framework of our stochastic formulation of fluid dynamics we shall now explain how to derive stochastic simulation schemes from our master equation. To begin with, we recall that the master equation contains all information about the possible transitions among the discrete states of the phase space  $\Gamma$  together with the corresponding transition rates. Since the general method of stochastic simulations has already been explained in a previous paper [8] we only present here the basic elements of this method.

The basic idea of the stochastic simulation method is to generate, by drawing random numbers, realizations of the stochastic process defined by the master equation. Such a realization will be denoted by  $(N_\omega(t), N_\psi(t))$ . Once an ensemble of realizations has been generated the quantity of interest can be obtained by evaluating the corresponding ensemble average.

In order to obtain a realization one repeats the following two steps until the desired final time is reached.

(1) *Determination of the stochastic time step.* Assuming that at a certain time  $t_0$  the state of the fluid is given by  $(N_\omega(t_0), N_\psi(t_0))$  a stochastic time step is to be determined from the total transition rate  $\Gamma_{\text{total}}$ , i.e. from the probability per unit time that a transition

occurs somewhere in the system. This quantity can be inferred from the master equation as the sum of all loss terms,

$$\Gamma_{\text{total}} = \Gamma_{\text{total}}(N_{\omega}(t_0), N_{\psi}(t_0)) = \sum_{\lambda\mu} \Gamma_{\lambda\mu} \quad (56)$$

where

$$\Gamma_{\lambda\mu} = \frac{4\nu}{\delta l^2} |N_{\omega}^{\lambda\mu}| + \frac{\delta\psi}{2\delta l^2} (|N^{\lambda\mu}| + |\mathcal{M}^{\lambda\mu}|) + \frac{1}{\varepsilon} |W^{\lambda\mu}|. \quad (57)$$

The stochastic time step  $\tau$  can then be obtained by drawing a random number  $\eta$  which is uniformly distributed over the interval  $[0, 1]$  and with the help of the formula

$$\tau = -\frac{1}{\Gamma_{\text{total}}} \ln \eta. \quad (58)$$

Note that this equation means that  $\tau$  is distributed exponentially and has mean value  $\langle \tau \rangle = 1/\Gamma_{\text{total}}$ .

(2) *Determination of the actual transition.* In order to determine the new state  $(N_{\omega}(t_0 + \tau), N_{\psi}(t_0 + \tau))$  of the fluid one first chooses a certain cell  $(\lambda, \mu)$  at random with probability  $\Gamma_{\lambda\mu}/\Gamma_{\text{total}}$ . Now, the actual transition is to be selected from 10 different possibilities. Each of these possibilities corresponds to one of the 10 terms in our master equation (four transitions correspond to the viscosity term  $\mathcal{A}_d$ , four to the convection term  $\mathcal{A}_c$ , and two to the constraint operator  $\mathcal{A}_p$ ). The correct relative transition probability is obtained by dividing the term that multiplies the shift operators by  $\Gamma_{\lambda\mu}$ . Note that in each transition exactly two of the stochastic integers change by the amount  $\pm 1$ .

Let us now illustrate the stochastic simulation method by performing some simulations for two-dimensional turbulence. The physical situation under study and the boundary conditions are precisely those explained in section 2 with  $L = 1$ . The initial conditions are defined as follows. The initial vorticity field is given by

$$\omega(x, 0) = \sum_k (B_k \cos kx + C_k \sin kx) \quad (59)$$

and the initial stream function is given by the constraint equation. The amplitudes  $B_k$  and  $C_k$  represent independent random Gaussian numbers with zero mean and variances

$$\langle B_k^2 \rangle = \langle C_k^2 \rangle = \langle \omega_k^* \omega_k \rangle|_{t=0} = \frac{\pi}{3k_0^2} \left( \frac{k}{k_0} \right)^2 \exp(-k/k_0). \quad (60)$$

These initial conditions guarantee that the initial fields are homogeneous and isotropic. Furthermore, we have as integral quantities the kinetic energy  $E$  and the enstrophy  $\Omega$ ,

$$E(t) = \frac{1}{2L^2} \int dx dy \langle u^2 + v^2 \rangle \quad \Omega(t) = \frac{1}{2L^2} \int dx dy \langle \omega^2 \rangle \quad (61)$$

and the palinstrophy  $P$  (enstrophy dissipation rate) which is defined by

$$\frac{\partial \Omega}{\partial t} = -2\nu P. \quad (62)$$

The initial condition may be characterized by the Reynolds numbers [21, 22]

$$R_L = \frac{E}{\nu(2\nu P)^{1/3}} \quad R_\lambda = \frac{\Omega^{3/2}}{2\nu P}. \tag{63}$$

It should be noted that the initial conditions as well as the Reynolds numbers and the resolution used in our stochastic simulations are similar to those used by Herring *et al* [21].

According to section 3 the initial conditions for the stochastic variables are given by

$$N_\omega^{\lambda\mu}(0) = \text{int} \left( \frac{\omega(\mathbf{x}_{\lambda\mu}, 0)}{\delta\omega} \right) \quad N_\psi^{\lambda\mu}(0) = \text{int} \left( \frac{\psi(\mathbf{x}_{\lambda\mu}, 0)}{\delta\psi} \right) \tag{64}$$

where  $\text{int}(\cdot)$  denotes the integer part. Within our discrete stochastic formulation the various quantities of interest are determined by taking appropriate ensemble averages. For example, the enstrophy is determined by the formula

$$\Omega(t) = \frac{1}{2} \sum_{\lambda\mu} \delta l^2 \langle \omega_{\lambda\mu}^2 \rangle \tag{65}$$

and the vorticity spectrum  $\langle \omega_k^* \omega_k \rangle$  is obtained by Fourier transforming the two-point correlation function which is defined to be

$$Q(\mathbf{x}_{\lambda\mu}, t) = \sum_{\lambda'\mu'} \langle \omega_{\lambda+\lambda', \mu+\mu'} \omega_{\lambda'\mu'} \rangle. \tag{66}$$

First we show in figures 1 and 2 one realization of the stochastic process defined by our master equation as it is obtained from a stochastic simulation with parameters  $\delta\omega = 10^{-3}$  and  $\delta l = 128^{-1}$ . The initial condition has been drawn from the initial ensemble defined above with  $k_0 = 4\pi$  and corresponds to the Reynolds numbers  $R_L(0) = 583$  and  $R_\lambda(0) = 31.7$ . We depict the stream function contours (figure 1) and the isovorticity lines (figure 2) at time  $t = 3$  which corresponds to approximately 3 turnover times of the large eddies. As one can see from the isovorticity lines in figure 2 the stochastic process  $\omega_{\lambda\mu}$  exhibits the basic features of evolving two-dimensional turbulence at early times, i.e. the formation of quasi-rectilinear vorticity gradient sheets [23]. It should be noted that the isovorticity lines displayed in figure 2 are rather choppy since the noise level in our simulation is relatively high and figure 2 shows only one realization of the underlying stochastic process.

In figures 3 and 4 we display the results which have been obtained by averaging over three realizations. The initial configurations have been drawn from the initial ensemble defined above with  $k_0 = 4\pi$ . The parameters are again  $\delta\omega = 10^{-3}$  and  $\delta l = 128^{-1}$  and the mean values of the initial Reynolds numbers are  $R_L(0) = 705$  and  $R_\lambda(0) = 35.2$ . Furthermore, we have the following initial values for energy, enstrophy and palinstrophy:

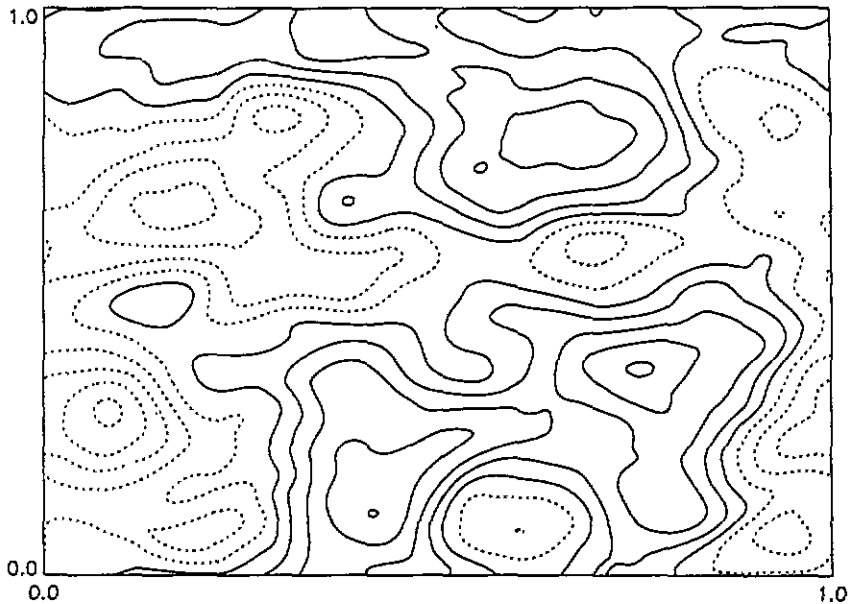
$$E(0) = 0.5042 \times 10^{-3} \quad \Omega(0) = 0.4937 \quad P(0) = 1477. \tag{67}$$

Figure 3 shows the palinstrophy (enstrophy dissipation rate)  $P$  as a function of time and, finally, we depict in figure 4 the energy spectrum

$$E_k = \frac{1}{4\pi L^2 k} \langle \omega_k^* \omega_k \rangle \tag{68}$$

obtained by averaging over the three realizations and over the three times  $t = 8$ ,  $t = 9$  and  $t = 10$ . As has been explained in section 3 the noise level induced by the random vorticity





**Figure 1.** One realization of the stochastic process  $\psi_{\lambda\mu} = \delta\psi N_{\psi}^{\lambda\mu}$  defined by the master equation (14). The initial condition has been drawn from the initial ensemble defined by equations (59) and (60). The figure shows the stream function contours at time  $t = 3$ . Parameters:  $\delta\omega = 10^{-3}$ ,  $\delta l = 128^{-1}$ ,  $k_0 = 4\pi$ . Reynolds numbers:  $R_L(0) = 583$ ,  $R_\lambda(0) = 31.7$ .

stress (48) has been subtracted in both figures in order to obtain the zero temperature quantities (see equation (54)) (actually, we have determined, from the expansion leading to the Hopf equation, the expression for the random vorticity stress to one order higher in  $\delta l$  than was done in section 4).

As is indicated in figure 4 the energy spectrum exhibits a power-law behaviour in the inertial range with a spectral exponent of approximately  $-3.5$ . This result is consistent with the results obtained by Brachet *et al* [23] which demonstrate that around the time of maximum enstrophy dissipation (see figure 3) the spectral exponent of the energy spectrum exhibits a transition from  $-4$  to  $-3$ . A more detailed investigation of the behaviour of the energy spectrum and, in particular, of the existence of a vorticity cascade [24, 25] leading to a spectral exponent of  $-3$  will be given elsewhere.

## 6. Conclusions

As is well known, Monte Carlo methods provide very efficient tools for the investigation of many-body systems in thermal equilibrium [26]. Within these methods one defines a probabilistic dynamic in such a way that the probability distribution relaxes in the long time limit to the equilibrium distribution which is known from statistical mechanics. However, since this probabilistic dynamic is, in general, very different from the true physical time evolution, the usual Monte Carlo methods are not appropriate for the study of the dynamical behaviour of systems far from equilibrium. It is this fact which has been the motivation of our stochastic formulation of fluid dynamics. In fact, the master equation representation of



Figure 2. One realization of the stochastic process  $\omega_{\lambda,\mu} = \delta\psi N_{\omega}^{\lambda,\mu}$  defined by the master equation (14). The initial condition has been drawn from the initial ensemble defined by equations (59) and (60). The figure shows the vorticity contours at time  $t = 3$ . The parameters are the same as in figure 1.

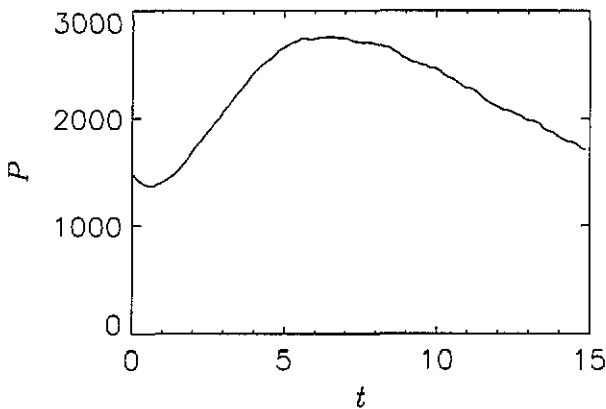


Figure 3. The palinstrophy  $P$  (enstrophy dissipation rate) as a function of time as it is obtained by averaging over three realization of the stochastic process defined by the master equation (14) and the initial ensemble defined by equations (59) and (60). The parameters are the same as in figure 1 and the mean values of the Reynolds numbers are:  $R_L(0) = 705$ ,  $R_\lambda(0) = 35.2$ .

the two-dimensional turbulence presented in this paper may be regarded as an example for the application of stochastic simulation methods to non-equilibrium systems.

Let us briefly summarize what has been achieved in this paper. The stochastic formulation of the two-dimensional Navier–Stokes equation is based on the multivariate master equation (14) which governs the probabilistic dynamics of a discrete stochastic

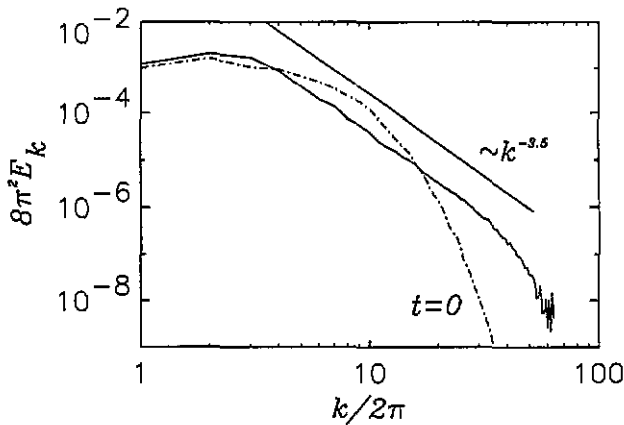


Figure 4. The energy spectrum  $E_k$  defined in equation (68) as it is obtained from a simulation of the stochastic process defined by the master equation (14) and the initial ensemble defined by equations (59) and (60) by averaging over three realizations and over the three times  $t = 8$ ,  $t = 9$ , and  $t = 10$ . The parameters are the same as in figure 1 and the mean values of the Reynolds numbers are:  $R_L(0) = 705$ ,  $R_\lambda(0) = 35.2$ .

process  $(N_\omega, N_\psi)$  representing the random vorticity and stream function. The applicability of our approach to the problem of two-dimensional turbulence follows from the equation of motion for the characteristic functional which has been derived from our master equation in section 4. It has been shown that this equation of motion leads to the Hopf functional equation of turbulence in the continuum limit and, thus, the whole hierarchy of turbulent correlation functions is contained in the stochastic process defined by the master equation. In other words, the master equation formulation presented in this paper leads to a stochastic simulation technique for the generating functional of the turbulent velocity field.

In order to understand more clearly the results of section 4, we draw again the analogy with Monte Carlo methods in equilibrium statistical mechanics. Any master equation which has been designed in order to evaluate, for example, the partition function of the system under study has to fulfil one basic property: it must be guaranteed that the relevant probability distribution which is known from equilibrium statistical mechanics represents a unique stationary solution of the master equation and that the initial states converge, in the limit of long times, to precisely this stationary solution. In comparison, the results of section 4 demonstrate that in the continuum limit the stochastic process defined by our master equation obeys the correct equations of motion which are conveniently expressed by the Hopf equation for the characteristic functional. It must be emphasized that, in contrast to the usual Monte Carlo methods, our approach is based on a representation of the full time evolution and does not require any information about the probability distribution of the turbulent fields.

On the basis of these theoretical results we have derived in section 5 a stochastic simulation algorithm which allows us to generate realizations of the stochastic process defined by the master equation. From a number of such realizations the interesting quantities are then obtained by evaluating the corresponding ensemble averages. This algorithm has been illustrated by performing some stochastic simulations of two-dimensional homogeneous turbulence. It has been demonstrated that the results provide a good description of the basic features of two-dimensional turbulence.

It should be evident that the stochastic approach presented in this paper may be

generalized to three space dimensions. Furthermore, one may include other equations of non-equilibrium thermodynamics such as the continuity equation for the mass and the balance equation for the internal energy. As we have already indicated, our stochastic formulation resembles closely the occupation number formalism which is used, for example, in the mesoscopic description of chemical reactions. It is therefore possible to extend our stochastic approach in a natural way to include the effect of chemical reactions and to study the dynamics of reacting flows.

## Appendix

This appendix is devoted to a more detailed analysis of the system of equations given by (1) and (5) by means of the method of elimination of fast variables. In particular, we shall derive an equation which makes possible to estimate the deviation of the fast variable  $\psi$  from the constraint manifold given by  $\Delta\psi + \omega = 0$ .

According to the general method the fast variable  $\psi$  is expanded in powers of  $\varepsilon$ ,

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots \quad (\text{A1})$$

Inserting this expansion into equation (5) we obtain to leading order  $\varepsilon^{-1}$ :

$$\Delta\psi_0 + \omega = 0. \quad (\text{A2})$$

This equation tells us that to leading order the fast variable  $\psi$  follows the slow variable  $\omega$  and that the constraint manifold is given by condition (3). The higher orders are easily found to yield:

$$\Delta\psi_1 = \dot{\psi}_0 + \nu\omega \quad (\text{A3})$$

and

$$\Delta\psi_{n+1} = \dot{\psi}_n \quad n = 1, 2, 3, \dots \quad (\text{A4})$$

By means of these equations  $\psi_{n+1}$  is determined recursively from  $\psi_n$ . In particular, we find

$$\Delta\psi_1 = -\Delta^{-1} \left[ \frac{\partial}{\partial x} \left( \psi_0 \frac{\partial \omega}{\partial y} \right) - \frac{\partial}{\partial y} \left( \psi_0 \frac{\partial \omega}{\partial x} \right) \right]. \quad (\text{A5})$$

Until now we have assumed that, in some sense,  $\varepsilon$  may be regarded as small. In order to formulate this assumption more precisely we now require

$$\|\varepsilon\Delta\psi_1\| \ll \|\Delta\psi_0\| \equiv \|\omega\| \quad (\text{A6})$$

where  $\|\cdot\|$  denotes some appropriate norm in the function space  $S$ . To be specific, we use in the following the  $L_2$  norm in the space of square integrable function on  $Q$ . In other words, we require that the first-order correction  $\varepsilon\Delta\psi_1$  is small compared with the leading order term  $\Delta\psi_0$ . It is easily seen that this requirement is equivalent to

$$\frac{\|\Delta\psi + \omega\|}{\|\omega\|} \ll 1. \quad (\text{A7})$$

A rough estimate is obtained by observing that we have in Fourier space

$$\Delta^{-1} = -1/k^2 \quad \omega \sim |k||v| \quad (\text{A8})$$

and that the nonlinearity of the vorticity equation may be approximated by

$$\left[ \frac{\partial}{\partial x} \left( \psi_0 \frac{\partial \omega}{\partial y} \right) - \frac{\partial}{\partial y} \left( \psi_0 \frac{\partial \omega}{\partial x} \right) \right]_k \sim k^2 v^2 \quad (\text{A9})$$

where the symbol  $\sim$  indicates that two quantities are of the same order of magnitude. Thus, we obtain from (A6)

$$\varepsilon \ll |k|/|v|. \quad (\text{A10})$$

The right-hand side may be estimated by taking for the wavenumber  $|k|$  its minimum value  $2\pi/L$  and for  $|v|$  the RMS velocity  $U := \langle v^2 \rangle^{1/2}$ . This finally yields

$$\varepsilon \ll 2\pi/U \cdot L. \quad (\text{A11})$$

This condition must be fulfilled in order to confine the fast variable  $\psi$  to the vicinity of the constraint manifold. For the stochastic simulations presented in section 5 condition (A11) gives  $\varepsilon \ll 190$ . In all simulations presented in this paper we have used  $\varepsilon = 10$  which was found to yields a relative error of

$$\|\Delta\psi + \omega\|/\|\omega\| < 5 \times 10^{-3} \quad (\text{A12})$$

for all times.

## References

- [1] Batchelor G K 1982 *The Theory of Homogeneous Turbulence* (Cambridge: Cambridge University Press)
- [2] McComb W D 1990 *The Physics of Fluid Turbulence* (Oxford: Clarendon)
- [3] Kraichnan R H and Montgomery D 1980 *Rep. Prog. Phys.* **43** 547
- [4] Rose H A and Sulem P L 1978 *J. Physique* **39** 441
- [5] Breuer H P and Petruccione F 1992 *Phys. Lett.* **172A** 49
- [6] Breuer H P and Petruccione F 1992 *Continuum Mech. Thermodyn.* **4** 247
- [7] Breuer H P and Petruccione F 1992 *J. Phys. A: Math. Gen.* **25** L661
- [8] Breuer H P and Petruccione F 1993 *Phys. Rev. E* **47** 1803
- [9] van Kampen N G 1981 1981 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)
- [10] Gardiner C W 1985 *Handbook of Stochastic Methods* 2nd edn (Berlin: Springer)
- [11] Hopf E 1952 *J. Rat. Mech. Anal.* **1** 87
- [12] Hopf E 1957 *Proc. Symp. Appl. Math. (American Mathematical Society)* **7** 41
- [13] Hopf E 1962 *Proc. Symp. Appl. Math. (American Mathematical Society)* **13** 157
- [14] Monin A S and Yaglom A M 1981 *Statistical Fluid Mechanics: Mechanics of Turbulence* vol 2 (Cambridge, MA: MIT)
- [15] Wolfram S 1986 *Theory and Applications of Cellular Automata* (Singapore: World Scientific)
- [16] Benzi R, Succi S and Vergassola M 1993 *Phys. Rep.* **222** 145
- [17] Breuer H P and Petruccione F 1993 *Physica* **192A** 569
- [18] Landau L D and Lifshitz E M 1959 *Fluid Mechanics* (London: Pergamon)
- [19] van Kampen N G 1985 *Phys. Rep.* **124** 69
- [20] Malek Mansour M, van den Broeck C, Nicolis G and Turner J W 1981 *Ann. Phys., NY* **131** 283
- [21] Herring J R, Orszag S A, Kraichnan R H and Fox D G 1974 *J. Fluid Mech.* **66** 417
- [22] Lilly D K 1969 *Phys. Fluids Suppl.* **II** 240
- [23] Brachet M E, Meneguzzi M, Politano H and Sulem P L 1988 *J. Fluid Mech.* **194** 333
- [24] Kraichnan R H 1967 *Phys., Fluids* **10** 1417
- [25] Batchelor G K 1969 *Phys. Fluids Suppl.* **II** 233
- [26] Binder K (ed) 1979 *Monte Carlo Methods in Statistical Physics* (Berlin: Springer)